On the Symmetric Gaussian Interference Channel with Partial Unidirectional Cooperation

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***Abstract*—A two-user symmetric Gaussian Interference Channel (IC) is considered in which a noiseless unidirectional link connects one encoder to the other. Having a constant capacity, the additional link provides partial cooperation between the encoders. It is shown that the available cooperation can dramatically increase the sum-capacity of the channel. This fact is proved based on comparison of proposed lower and upper bounds on the sum-capacity. Partitioning the data into three independent messages, namely private, common, and cooperative ones, the transmission strategy used to obtain the lower bound enjoys a simple type of Han-Kobayashi scheme together with a cooperative communication scheme. A Genie-aided upper bound is developed which incorporates the capacity of the cooperative link. Other upper bounds are based on the sum-capacity of the Cognitive Radio Channel and cut-set bounds. For the strong interference regime, the achievablity scheme is simplified to employ common and/or cooperative messages but not the private one. Through a careful analysis it is shown that the gap between these bounds is at most one and two bits per real dimension for strong and weak interference regimes, respectively. Moreover, the Generalized Degrees-of-Freedom of the channel is characterized.**

I. Introduction

Interference is one of the major limiting factors in achieving the total throughput of a network consisting of multiple non-cooperative transmitters intending to convey independent messages to their corresponding receivers through a common bandwidth. The way interference is usually dealt with is either by treating it as noise or preventing it by associating different orthogonal dimensions, e.g. time or frequency division, to different users. Since interference has structure, it is possible for a receiver to decode some part of the interference and remove it from the received signal. This is, in fact, the coding scheme proposed by Han-Kobayashi (HK) for the two-user Gaussian IC [1]. The two-user Gaussian IC provides a simple example showing that a single strategy against interference is not optimal. In fact, one needs to adjust the strategy according to the channel parameters [2]–[4]. However, a single suboptimal strategy can be proposed to achieve up to a single bit per user of the capacity region of the two-user Gaussian IC [5].

If the senders can cooperate, interference management can be done more effectively through cooperation. Cooperative links can be either orthogonal (using out-of-band signalling) or non-orthogonal (using in-band signalling) to the shared medium [6]. In this work, unidirectional orthogonal cooperation is considered.

Prior Works. Transmitter coordination over orthogonal links is considered in different scenarios using two-transmitter configurations as indicated in [6]–[13]. The capacity region of the Multiple Access Channel (MAC) with bidirectional cooperation is derived in [7], where cooperation is referred to as *conference*. Several achievable rate regions are proposed for the IC with bidirectional transmitter and receiver cooperation [8]. The transmit cooperation employs Dirty Paper Coding (DPC), whereas the receive cooperation uses Wyner-Ziv coding [14]. In the transmit cooperation, the entire message of each transmitter is decoded by the other cooperating transmitter, which apparently limits the performance of the scheme to the capacity of the cooperative link. The capacity regions of the compound MAC with bidirectional transmitter cooperation and the IC with Unidirectional Cooperation (ICUC) under certain strong interference conditions are obtained in [9]. These rate regions are shown to be closely related to the capacity region of the Compound MAC with Common Information (CMACCI). Furthermore, the inner and outer bounds for the ICUC considered in [9] are given in [10] for different interference regimes. The above papers considered the ICUC in which the message sent by one of the encoders is *completely* known to the other encoder. This setup is referred to as *cognitive radio* in the literature and is also considered in [11]–[13]. More recently, the capacity region of the compound MAC with both encoder and decoder unidirectional cooperation, and physically degraded channels is derived in [6]. In a related problem, the sum-capacity of the ICUC channel with a different transmission model is characterized upto 9 bits [15].

Contribution. For a Symmetric ICUC (SICUC) setup shown in Fig. 1 when $aP > 1$, a simple HK scheme in conjunction with cooperative communication is employed in the weak interference regime, *i.e.*, $a \leq 1$. In particular, each user partitions its data into three independent messages, namely private, common and cooperative, and constructs independent codebooks for each of them. Each receiver, first jointly decodes the common and cooperative messages of both users and then decodes its own private message. Following [5], the power of the private message of each user is set such that it is received at the level of the Gaussian noise at the other receiver. This is to guarantee that the interference caused by the private message has a small effect on the other user. To prove that the scheme has negligible gap from the sum-capacity for *all* transmit powers and channel gains, the transmit

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power should be properly allocated among the messages used by each user. Optimizing the achievable sum-rate requires finding an optimum Power Allocation (PA) over different codebooks employed by each user, which is mathematically involved. For the weak interference regime, a simple power allocation strategy that works for both users and operates over the three codebooks is then proposed. Noting that the private codebook contributes in a small rate for the strong interference regime, *i.e.*, $a > 1$, the scheme is subsequently modified to achieve a smaller gap and complexity by utilizing one or both of the common and cooperative codebooks but not the private codebook. The achievable sum-rates are compared to appropriate upper bounds to prove that the gap from the sum-capacity of the channel is one and two bits per real dimension, for the strong and weak interference regimes, respectively. The upper bounds are based on the cut-set bound and the sum-capacity of the Cognitive Radio Channel (CgRC), in which the message sent by one of the encoders is known to the other (cooperating) encoder. A new upper bound is also proposed to incorporate the effect of the imperfect cooperative link, *i.e.*, the case that only part of the message sent by one of the encoders is known to the other (cooperating) encoder [16]. To obtain some asymptotic results, the Generalized Degrees of Freedom (GDOF) of the channel is characterized. The GDOF represents the number of dimensions available for communication in the presence of the interference when Signal-to-Noise Ratio SNR $\to \infty$. When SICUC is noise-limited ($aP \leq 1$), it is shown that treating interference as noise, and not using the cooperative link, gives a sum-rate within a single bit to the sum-capacity of the channel for all channel parameters.

The rest of this paper is organized as follows: Section II presents the system model. Sections III and IV, respectively, focus on the weak and strong interference regimes. For each regime, the corresponding achievable sum-rate, power allocation, upper bounds, and the gap analysis are provided. Section V characterizes the GDOF of the channel. Section VI considers the noise-limited regime. Finally, section VII concludes the paper.

Notation. The sequence $\mathbf{x}^n$ denotes $x_1, \cdots, x_n$. $\bar{x} \triangleq 1-x$. $\text{Prob}(\cdot)$ indicates the probability of an event, and $\mathbb{E}[X]$ represents the expectation over the random variable X. All logarithms are to the base 2 and all rates are expressed in bits per real dimension. Finally, $C(P) \triangleq \frac{1}{2}\log(1+P)$.

II. System Model

In this work, a two-user symmetric Gaussian interference channel with partial unidirectional cooperation, as depicted in Fig. 1, is considered. The model consists of two transmitter-receiver pairs, in which each transmitter wishes to convey its own data to its corresponding receiver. There exists a noiseless cooperative link with capacity C_{12} from encoder 1 to encoder 2. It is assumed that all nodes are equipped with a single antenna. The input-output relationship for this channel in standard form is expressed as [17]:

$$\begin{aligned} y_1 &= x_1 + \sqrt{a}\ x_2 + z_1, \\ y_2 &= \sqrt{a}\ x_1 + x_2 + z_2, \end{aligned} \tag{1}$$

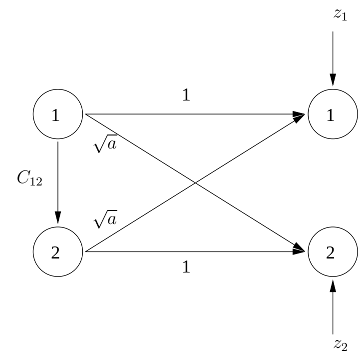

Fig. 1. The symmetric interference channel with unidirectional transmitter cooperation.

where constant $a \geq 0$ represents the gain of the interference links. For $i \in \{1,2\}$, $z_i \sim \mathcal{N}(0,1)$. The average power constraint of each transmitter is P, *i.e.*, $\mathbb{E}[|x_i^2|] \leq P$. The full Channel State Information (CSI) is assumed to be available at the transmitters as well as receivers. Following [5], $\text{SNR}=P$ and Interference-to-Noise Ratio $\text{INR}=aP$ are defined as the characterizing parameters of SICUC.

For a given block length n, encoder i sends its own (random) message index m_i from the index set $\mathcal{M}_i = \{1,2,...,M_i = 2^{nR_i}\}$ with rate R_i [bits/channel use]. Each pair (m_1, m_2) occurs with the same probability $1/M_1M_2$. A one-step conference is made up of a communicating function k which maps the message index m_1 into $\mathbf{q}^n$ with finite alphabet size $|\mathcal{Q}| = 2^{nC_{12}}$. The encoding function f_1 maps the message index m_1 into a codeword $\mathbf{x}_1^n$ chosen from the codebook $\mathcal{C}_1$. The encoding function f_2 maps the message index m_2 and what was obtained from the conference with encoder 1 into a codeword $\mathbf{x}_2^n$ selected from the codebook $\mathcal{C}_2$. Therefore:

$$\begin{aligned} \mathbf{q}^n &= k(m_1), \\ \mathbf{x}_1^n &= f_1(m_1), \\ \mathbf{x}_2^n &= f_2(m_2, \mathbf{q}^n). \end{aligned} \tag{2}$$

The codewords in each codebook must satisfy the average power constraint $\frac{1}{n}\sum_{t=1}^{n}|x_{i,t}|^2 \leq P$ for $i \in \{1,2\}$. Each decoder uses a decoding function $g_i(\mathbf{y}_i^n)$ to decode its desired message index m_i based on its received sequence. Let $\hat{m}_i$ be the output of the decoder. The average probability of error for each decoder is: $P_{e_i} = \mathbb{E}[\text{Prob}(\hat{m}_i \neq m_i)]$. A rate pair $(R_1,\ R_2)$ is said to be achievable when there exists an $(M_1, M_2, n, P_{e_1}, P_{e_2})$-code for the ICUC consisting of two encoding functions $\{f_1, f_2\}$ and two decoding functions $\{g_1, g_2\}$ such that for sufficiently large n:

$$\begin{aligned} R_1 &\leq \tfrac{1}{n}\log(M_1), \\ R_2 &\leq \tfrac{1}{n}\log(M_2), \\ P_e &\leq \epsilon. \end{aligned}$$

In the above, $P_e = \max(P_{e_1}, P_{e_2})$ and $\epsilon \geq 0$ is a constant that can be chosen arbitrarily small. The capacity region of the ICUC is the closure of the set of achievable rate pairs. In this work, the characterization of the sum-capacity of the channel is considered. In addition, the interference-limited regime is mainly investigated, *i.e.*, $\text{INR} \geq 1$ since otherwise the system is noise limited and is not of much interest [5], [16]. In fact,

it will be shown later that treating interference as noise, and not using the cooperative link, will give a sum-rate within a single bit to the sum-capacity of the channel for all channel parameters. Finally, for rate derivations in this paper, Gaussian codebooks are employed.

III. Weak Interference Regime ($a \leq 1$)

A. Achievable Sum-Rate

A simple type of the Han-Kobayashi scheme is used in [5] for the symmetric IC, in which the codebook of transmitter i is composed of private and common codebooks denoted by $\mathcal{C}_i^u$ and $\mathcal{C}_i^w$, respectively. The transmitted signal is $x_i = \sqrt{P_u}u_i + \sqrt{P_w}w_i$, and it is assumed that $\mathbb{E}[|u_i|^2] = \mathbb{E}[|w_i|^2] = 1$. The power associated with the codebooks are P_u and P_w, such that $P_u + P_w = P$. First, the common messages are decoded at both decoders and their effect is subtracted from the received signal. Then, the private messages are decoded at their corresponding receiver while treating the private signal of the other user as noise. One of the main steps towards getting a small gap is that the power of the interfering private message is set to be at the noise level, *i.e.*, $aP_u = 1$ [5].

In this work, because the existence of the cooperative link, encoder 2 can relay the transmitter 1's information to help the other receiver [18]. Therefore, it is natural to have three codebooks at encoder i, namely private codebook $\mathcal{C}_i^u$ with codewords u_i, common codebook $\mathcal{C}_i^w$ with codewords w_i , and cooperative codebook $\mathcal{C}_i^v$ with codewords v. The cooperative message is decoded error free at transmitter 2 and relayed to both receivers. The input to the channel can be written as:

$$X_i = \sqrt{P_u}u_i + \sqrt{P_{w_i}}w_i + \sqrt{P_{v_i}}v. \tag{3}$$

The average power constraint for transmitter i is $P_u + P_{w_i} + P_{v_i} = P$. The following power allocation is used:

$$\begin{array}{ll} P_u & = \frac{1}{a}, \\ P_{v_i} & = \gamma_i(P - \frac{1}{a}), \\ P_{w_i} & = \bar{\gamma}_i(P - \frac{1}{a}), \end{array} \tag{4}$$

where $0 \leq \gamma_i \leq 1$ is the corresponding power allocation parameter for transmitter i. The received power of v at decoder 1 is: $P_V = (\sqrt{P_{v_1}} + \sqrt{aP_{v_2}})^2$.

Because of the symmetries in the problem (in terms of the channel gains), and the fact that the sum-rate is the objective of this paper, we set $\gamma_1 = \gamma_2 = \gamma$. It is also assumed that v is decoded at both decoders. Therefore, the achievable rate pairs associated with the HK scheme are in fact the intersection of the capacity regions of two MACs: each composed of four virtual users, *i.e.*, $\{u_1, w_1, w_2, v\}$ for MAC$_1$, and $\{u_2, w_1, w_2, v\}$ for MAC$_2$. Among all rate assignments and decoding orders for the MACs, the following scheme is used. It will be shown later that the scheme achieves within two bits of the associated upper bound. The strategy is along the same line as the approach of [5] for the IC. The common codewords w_1, w_2, and the cooperative codeword v are jointly decoded at both receivers, treating the interfering private signals as noise. Finally, each receiver decodes its private codeword[1]. Therefore, the capacity region of four-user MACs are projected onto three-dimensional subspaces. As a result, a compound MAC with three virtual users (w_1, w_2, v) is obtained. The capacity region of MAC$_i$ $i \in \{1, 2\}$ is:

$$R_v \leq \min\{C(\frac{P_V}{P_u+2}), C_{12}\}, \tag{5}$$

$$R_{w_i} \leq C(\frac{P_w}{P_u+2}), \tag{6}$$

$$R_{w_j} \leq C(\frac{aP_w}{P_u+2}), \tag{7}$$

$$R_{w_i} + R_{w_j} \leq C(\frac{P_w(1+a)}{P_u+2}), \tag{8}$$

$$R_{w_i} + R_v \leq C(\frac{P_w + P_V}{P_u+2}), \tag{9}$$

$$R_{w_j} + R_v \leq C(\frac{aP_w + P_V}{P_u+2}), \tag{10}$$

$$R_{w_i} + R_{w_j} + R_v \leq C(\frac{P_w + aP_w + P_V}{P_u+2}), \tag{11}$$

where $j \in \{1, 2\}$, and $j \neq i$. Also, $P_{w_1} = P_{w_2} \triangleq P_w$, $P_{v_1} = P_{v_2} \triangleq P_v$, and

$$P_V = (1 + \sqrt{a})^2 P_v. \tag{12}$$

The factor 2 in the denominators comes from the fact that the power of the interfering private signal has been set to unity. It is also assumed that: $R_{w_1} = R_{w_2} \triangleq R_w$. Since the sum-rate is $2(R_u + R_w) + R_v$, in the above, $2R_w + R_v$ is of particular interest. There exist three ways to construct $2R_w + R_v$ from inequalities (5-11):

$$2R_w + R_v \leq C(\frac{P_w(1+a) + P_V}{P_u+2}) \triangleq R_{\mathcal{B}_1}, \tag{13}$$

$$2R_w + R_v \leq 2\tilde{R}_w + \min\{C(\frac{P_V}{P_u+2}), C_{12}\} \triangleq R_{\mathcal{B}_2}, \tag{14}$$

$$2R_w + R_v \leq C(\frac{aP_w + P_V}{P_u+2}) + \tilde{R}_w \triangleq R_{\mathcal{B}_3}, \tag{15}$$

where

$$\tilde{R}_w \triangleq \min\{C(\frac{aP_w}{P_u+2}), \frac{1}{2}C(\frac{P_w(1+a)}{P_u+2})\}, \tag{16}$$

and (13-15) come from inequalities (11), (5-7), and (6, 7, 9, 10), respectively. The achievable sum-rate is:

$$R_{\text{sum}} = \max_{0 \leq \gamma \leq 1} \{2C(\frac{P_u}{2}) + \min_{i \in \{1,2,3\}} \{R_{\mathcal{B}_i}\}\}. \tag{17}$$

Optimizing the achievable sum-rate requires finding optimum power allocation parameter γ, which is mathematically involved. In the following, a simple power allocation strategy is provided, and is later shown to be within two bits of the related upper bound.

[1]It is remarked that encoder 2 could perform the Gelfand-Pinsker binning [19] in order to precode its own message against the known interference. This, in general, could increase the rate at receiver 2, especially in the weak interference regime. In this paper, however we show that one can achieve a sum-rate within two bits of the sum-capacity of SICUC without binning. Employing the binning technique will be considered in future works.

B. Power Allocation Policies and the Associated Sum-Rates

1) Universal PA: As will be shown later, the following power allocation guarantees a small gap form the sum-capacity for all $a \leq 1$ and hence called *universal*. We set $P_w = P_V$, which together with $P_u = \frac{1}{a}$ give:

$$P_w = \frac{P - \frac{1}{a}}{1 + \frac{1}{(1+\sqrt{a})^2}}. \tag{18}$$

The corresponding sum-rate is $R + 2C(\frac{P_u}{2})$, where

$$R \triangleq \min_{i \in \{1, \cdots, 5\}} \{R_i\}, \tag{19}$$

and

$$\begin{aligned} R_1 &= C(\frac{(2+a)P_w}{P_u+2}), \\ R_2 &= 2C(\frac{aP_w}{P_u+2}) + C_{12}, \\ R_3 &= C(\frac{(1+a)P_w}{P_u+2}) + C_{12}, \\ R_4 &= C(\frac{(1+a)P_w}{P_u+2}) + C(\frac{aP_w}{P_u+2}), \\ R_5 &= \frac{3}{2}C(\frac{(1+a)P_w}{P_u+2}) \end{aligned} \tag{20}$$

are obtained from inequalities (13-15), Eq. (18). Note that the following redundant inequalities are eliminated:

$$\begin{aligned} R_6 &= 2C(\frac{aP_w}{P_u+2}) + C(\frac{P_w}{P_u+2}), \\ R_7 &= C(\frac{(1+a)P_w}{P_u+2}) + C(\frac{P_w}{P_u+2}). \end{aligned}$$

It is remarked that this allocation achieves the optimal GDOF of the channel as will be shown later and therefore can be among the achievable schemes that have a small gap from the sum-capacity for all channel parameters[2]. This allocation also simplifies the gap analysis.

2) Full Cooperation PA: It is also interesting to consider the case that all the remaining $P - \frac{1}{a}$ power is devoted to cooperation or equivalently $P_w = 0$, *i.e.*, $\gamma = 1$ in (4). In this case, the achievable sum-rate becomes:

$$R_{\text{FC}} = \min\{C_{12}, C(\frac{(1+\sqrt{a})^2(aP-1)}{2a+1})\} + 2C(\frac{P_u}{2}). \tag{21}$$

It will be shown that this power allocation provides a smaller gap compared to the universal PA in some regions. However, it does not achieve the GDOF *universally* [16], and hence cannot provide a small gap for all channel parameters.

C. Upper Bounds

Two upper bounds are used for this regime. The first one is an enlarged version of the sum-capacity of the Cognitive Radio Channel (CgRC) with weak interference [12]. The sum-capacity of the CgRC for $a \leq 1$, denoted by C^w_{sum}, is [12]:

$$C^w_{\text{sum}} = \frac{1}{2} \max_{0 \leq \eta \leq 1} \left[\log(1 + aP + 2\sqrt{\bar{\eta} a}P + P) + \log(\frac{1+\eta P}{1+\eta aP}) \right].$$

[2]In fact, if an achievable scheme fails to achieve the GDOF of the channel it cannot have a small gap from the sum-capacity for all channel parameters.

It is noted that the first and the second terms in the above upper bound are monotonically decreasing and increasing functions of η, respectively. The above upper bound is enlarged by setting $\bar{\eta} = 1$ and $\eta = 1$ in the first and second terms, respectively. Therefore, the following upper bound serves as the first upper bound for the region:

$$R^{(1)}_{\text{ub}} = \frac{1}{2} \log\left(1 + (1+\sqrt{a})^2 P\right) + \frac{1}{2} \log(\frac{1+P}{1+aP}). \tag{22}$$

For the IC ($C_{12} = 0$), Etkin, *et.al* [5] provide a new upper bound by selecting an appropriate genie. In the following, the effect of the cooperative link is incorporated in the genie-aided upper bound. The genie signal $s_i,\ i = 1, 2$ is provided to receiver i:

$$\begin{aligned} s_1 &= \sqrt{a} x_1 + z_2, \\ s_2 &= \sqrt{a} x_2 + z_1. \end{aligned} \tag{23}$$

The upper bound on the sum-capacity of the genie-aided channel is:

$$\begin{aligned} nR_{\text{sum}} &\leq I(m_1; \mathbf{y}^n_1, \mathbf{s}^n_1, \mathbf{q}^n) + I(m_2; \mathbf{y}^n_2, \mathbf{s}^n_2, \mathbf{q}^n) + n\epsilon_n \\ &\overset{\text{(a)}}{=} I(m_1; \mathbf{q}^n) + I(m_1; \mathbf{y}^n_1, \mathbf{s}^n_1 | \mathbf{q}^n) + I(m_2; \mathbf{q}^n) \\ &\quad + I(m_2; \mathbf{y}^n_2, \mathbf{s}^n_2 | \mathbf{q}^n) + n\epsilon_n \\ &\overset{\text{(b)}}{=} h(\mathbf{q}^n) + I(m_1; \mathbf{y}^n_1, \mathbf{s}^n_1 | \mathbf{q}^n) + \\ &\quad I(m_2; \mathbf{y}^n_2, \mathbf{s}^n_2 | \mathbf{q}^n) + n\epsilon_n \\ &\leq \sum_{t=1}^{n} h(q_t) + I(m_1; \mathbf{s}^n_1 | \mathbf{q}^n) + I(m_1; \mathbf{y}^n_1 | \mathbf{s}^n_1, \mathbf{q}^n) \\ &\quad + I(m_2; \mathbf{s}^n_2 | \mathbf{q}^n) + I(m_2; \mathbf{y}^n_2 | \mathbf{s}^n_2, \mathbf{q}^n) + n\epsilon_n \\ &= nC_{12} + h(\mathbf{s}^n_1 | \mathbf{q}^n) - h(\mathbf{s}^n_1 | m_1, \mathbf{q}^n) + \\ &\quad I(m_1; \mathbf{y}^n_1 | \mathbf{s}^n_1, \mathbf{q}^n) + h(\mathbf{s}^n_2 | \mathbf{q}^n) - h(\mathbf{s}^n_2 | m_2, \mathbf{q}^n) + \\ &\quad I(m_2; \mathbf{y}^n_2 | \mathbf{s}^n_2, \mathbf{q}^n) + n\epsilon_n \\ &\overset{\text{(c)}}{=} nC_{12} + h(\mathbf{s}^n_1 | \mathbf{q}^n) - h(\mathbf{s}^n_1 | \mathbf{x}^n_1) + \\ &\quad h(\mathbf{y}^n_1 | \mathbf{s}^n_1, \mathbf{q}^n) - h(\mathbf{y}^n_1 | \mathbf{s}^n_1, \mathbf{q}^n, m_1, \mathbf{x}^n_1(m_1)) + \\ &\quad h(\mathbf{s}^n_2 | \mathbf{q}^n) - h(\mathbf{s}^n_2 | \mathbf{x}^n_2) + h(\mathbf{y}^n_2 | \mathbf{s}^n_2, \mathbf{q}^n) \\ &\quad - h(\mathbf{y}^n_2 | \mathbf{s}^n_2, \mathbf{q}^n, m_2, \mathbf{x}^n_2(m_2, \mathbf{q}^n)) + n\epsilon_n \\ &\overset{\text{(d)}}{\leq} nC_{12} + h(\mathbf{s}^n_1 | \mathbf{q}^n) - h(\mathbf{z}^n_2) + h(\mathbf{y}^n_1 | \mathbf{s}^n_1) \\ &\quad - h(\mathbf{y}^n_1 | \mathbf{s}^n_1, \mathbf{q}^n, m_1, \mathbf{x}^n_1(m_1)) + h(\mathbf{s}^n_2 | \mathbf{q}^n) \\ &\quad - h(\mathbf{z}^n_1) + h(\mathbf{y}^n_2 | \mathbf{s}^n_2) \\ &\quad - h(\mathbf{y}^n_2 | \mathbf{s}^n_2, \mathbf{q}^n, m_2, \mathbf{x}^n_2(m_2, \mathbf{q}^n)) + n\epsilon_n \\ &\overset{\text{(e)}}{=} nC_{12} - h(\mathbf{z}^n_1) - h(\mathbf{z}^n_2) + h(\mathbf{y}^n_1 | \mathbf{s}^n_1) + \\ &\quad h(\mathbf{y}^n_2 | \mathbf{s}^n_2) + n\epsilon_n, \end{aligned} \tag{24}$$

where (a), (b), and (c) respectively, follow from the chain rule of mutual information [18], Eq. (2), and Markovity of $m_1 \to \mathbf{x}^n_1 \to \mathbf{s}^n_1$ and $(\mathbf{q}^n, m_2) \to \mathbf{x}^n_2 \to \mathbf{s}^n_2$. (d) comes from Eq. (23) and the fact that removing the condition $\mathbf{q}^n$ does not decrease the entropy. (e) is valid because $\mathbf{y}^n_2$ and $\mathbf{s}^n_2$ are independent conditionally on $(\mathbf{x}^n_2, \mathbf{q}^n)$. The same fact is applied to $\mathbf{y}^n_1$ and $\mathbf{s}^n_1$ conditionally on $(\mathbf{x}^n_1, \mathbf{q}^n)$. Following the same reasoning as [5] for $h(\mathbf{y}^n_i | \mathbf{s}^n_i),\ i \in \{1, 2\}$, we get the following upper bound:

$$R^{(2)}_{\text{ub}} = C_{12} + \log\left(\frac{P + (aP+1)^2}{aP+1}\right). \tag{25}$$

We remark that the above upper bound is the same as the upper bound derived in [5], except for the additional term C_{12}.

D. Gap Analysis

Theorem 1 For $a \leq 1$, the achievable sum-rate associated with the universal power allocation policy of (18) is within two bits of the sum-capacity of the channel.

Proof: See Appendix A. ■

In the next theorem, the maximum gap is shown to be smaller than 2 bits using full cooperation for some cases.

Theorem 2 The achievable sum-rate corresponding to the full cooperation power allocation policy (Eq. (21)) is within 1 and 1.5 bits of the sum-capacity of the channel for the following cases, respectively.

1) CASE I: $C_{12} \leq C(\frac{b(aP-1)}{2a+1})$ and $a^3P^2 \leq a+1$
2) CASE II: $C_{12} \geq C(\frac{b(aP-1)}{2a+1})$,

where
$$b \triangleq (1+\sqrt{a})^2, \tag{26}$$

Proof: For each case, the gap Δ_{FC} is calculated:

1) CASE I: The achievable sum-rate is:
$$R_{\text{sum}} = C_{12} + 2C(\frac{1}{2a}).$$
The gap from the second upper bound, *i.e.*, Eq. (25) is:
$$\Delta_{\text{FC,I}} = 1 + \log\left(1 + \frac{a^3P^2 - a - 1}{(aP+1)(2a+1)}\right) \leq 1.$$
2) CASE II: In this case, the achievable sum-rate
$$R_{\text{sum}} = \frac{1}{2}\Big[\log\big(\frac{2a+1+b(aP-1)}{a}\big) + \log\big(\frac{2a+1}{a}\big)\Big] - 1$$
is compared to the upper bound of Eq. (22):
$$\begin{aligned}\Delta_{\text{FC,II}} &= 1 + \frac{1}{2}\log\left(\frac{a+aP}{(2a+1)(1+aP)}\right) + \\ &\quad + C\left(\frac{2\sqrt{a}}{2a+1+b(aP-1)}\right) \\ &\leq 1 + C\left(\frac{2\sqrt{a}}{a+1}\right) \\ &\leq \frac{3}{2}.\end{aligned}$$
■

IV. Strong Interference Regime ($a > 1$)

A. Achievable Sum-Rate

It is remarked that for $a \geq 1$, there is little benefit using the private codebooks because: $\log(1+\frac{P_u}{2}) = \log(1+\frac{1}{2a}) \leq 1$. Therefore, we can modify the achievablity scheme to have only w_1, w_2, and v. Here, we propose simple PA policies for the following cases and prove that the gap between the corresponding achievable rates and sum-capacity is less than 1 bit for each scenario. For $a \geq 1$, three cases may happen:

1) $P \leq 1 \leq a$
2) $1 \leq a \leq P$
3) $1 \leq P \leq a$

For the first case, user 2 can at most get $C(P) \leq 1$. Since $a > 1$ we can simply use all the available power for cooperation *i.e.*, $P_w = P_u = 0$.

The common codebooks play an important role for the last two cases in the IC without the cooperative link. As IC is a special case of ICUC with $C_{12} = 0$, we give most of the available power to common codewords in these cases.

1) $P \leq 1 \leq a$: In this case, we let $P_w = P_u = 0$ and $P_V = bP$. The achievable sum-rate becomes:
$$R_{\text{sum}} = \min\{C_{12}, C(bP)\}. \tag{27}$$

2) $1 \leq a \leq P$: In this case, let $P_w = P$ and $P_u = P_V = 0$. It is straightforward to show that $C\big((1+a)P_w\big) \leq 2C(P_w)$. Therefore, the achievable sum-rate becomes:
$$R_{\text{sum}} = C\big((1+a)P\big). \tag{28}$$

3) $1 \leq P \leq a$: In this scenario, it is suggested that receiver 1 jointly decodes w_2, v first and then w_1. Similarly, receiver 2 jointly decodes w_1, v first and then w_2. The following power allocation is used:
$$\begin{aligned} P_w &= P - 1, \\ P_V &= b. \end{aligned} \tag{29}$$
The rate constraints are:
$$\begin{aligned} R_v &\leq \min\{C_{12}, C(\frac{P_V}{P_w+1})\}, \\ R_w &\leq \min\{C(P_w), C(\frac{aP_w}{P_w+1})\}, \\ R_v + R_w &\leq C(\frac{P_V + aP_w}{P_w+1}). \end{aligned}$$
Since $a \geq P$, the power allocation leads to:
$$R_w = \min\{C(P-1), C(\frac{a(P-1)}{P})\} = C(P-1).$$
Therefore, the achievable sum-rate becomes:
$$\begin{aligned} R_{\text{sum}} &= \min\Big\{\min\big\{C_{12}, C(\tfrac{b}{P})\big\} + \\ &\quad + C(P-1), C(\tfrac{aP+1+2\sqrt{a}}{P})\Big\} + C(P-1). \end{aligned} \tag{30}$$

B. Upper Bounds

Using the cut-set analysis [18], transmitter 1 can send at most up to $C_{12} + C(P)$ bits per channel use. It is clear that transmitter 2 can provide at most $C(P)$ bits per channel use to its corresponding receiver. Another upper bound is the sum-capacity of the CgRC for the specific strong interference regime reported in [9]. It is easy to verify that the SICUC with $a \geq 1$ satisfies the strong interference conditions of [9]. Therefore, the sum-rate is upper bounded by:
$$R_{\text{ub}}^{(3)} = \min\{C_{12} + 2C(P), C(bP)\}. \tag{31}$$

C. Gap Analysis

1) $P \leq 1 \leq a$: In this case, if $C_{12} \leq C(bP)$, the achievable sum-rate of Eq. (27) is compared to the first term of the upper bound of Eq. (31), otherwise it is compared to the second term of Eq. (31). Therefore the gap $\Delta \leq 2C(P) \leq 1$.

2) $1 \leq a \leq P$: In this case, the difference (Δ) between the second term in the upper bound of Eq. (31) and the achievable sum-rate of Eq. (28) is calculated as follows:

$$\begin{aligned}\Delta &= \frac{1}{2}\log\big(1+\frac{2\sqrt{a}P}{1+aP+P}\big)\\ &\leq \frac{1}{2}\log\big(1+\frac{aP+P}{1+aP+P}\big)\\ &\leq \frac{1}{2}.\end{aligned}$$

3) $1 \leq P \leq a$: Because of Eq. (30) two cases can occur:

- $C_{12} > C\big(\frac{b}{P}\big)$: For this condition, the achievable sum-rate of Eq. (30) is:

$$\begin{aligned}R_{\text{sum}} &= \frac{1}{2}\min\big\{\log(P^2+bP), \log(P+aP+1+2\sqrt{a})\big\}\\ &= \frac{1}{2}\log(P+aP+1+2\sqrt{a}).\end{aligned}$$

  Then the gap with respect to Eq. (31) becomes:

$$\begin{aligned}\Delta &= \frac{1}{2}\log(1+\frac{2\sqrt{a}(P-1)}{2\sqrt{a}+1+aP+P})\\ &\leq \frac{1}{2}\log(1+\frac{2\sqrt{a}P}{1+aP+P})\\ &\leq \frac{1}{2}\log(1+\frac{aP+P}{1+aP+P})\\ &\leq \frac{1}{2}.\end{aligned}$$

- $C_{12} \leq C\big(\frac{b}{P}\big)$: In this case, the gap between the following achievable sum-rate:

$$R_{\text{sum}} = \min\left\{C_{12}+\log(P), \frac{1}{2}\log(1+P+aP+2\sqrt{a})\right\}$$

  and the upper bound of Eq. (31) is:

$$\begin{aligned}\Delta &\leq \max\left\{\log(1+\tfrac{1}{P}), \tfrac{1}{2}\log(1+\tfrac{2\sqrt{a}(P-1)}{1+P+aP+2\sqrt{a}})\right\}\\ &\leq 1.\end{aligned}$$

D. Sum-Capacity for Sufficiently Large C_{12}

In [16], we simply showed that if

$$C_{12} \geq C\big(bP\big), \tag{32}$$

by setting $P_{v_1} = P_{v_2} = P$, we can achieve the second term of Eq. (31) and hence, the sum-capacity of the channel.

V. GDOF Analysis

In this section, the sum-capacity behavior is considered in the high SNR regime by characterizing the GDOF of the channel.

It is known that the interference can reduce the available degrees of freedom for data communication [5]. To understand this effect, we define the GDOF as below:

$$d(\alpha,\beta) = \lim_{\text{SNR}\to\infty}\frac{R_{\text{sum}}(\text{SNR},\alpha,\beta)}{C(\text{SNR})}, \tag{33}$$

where

$$\alpha \triangleq \frac{\log(\text{INR})}{\log(\text{SNR})}, \tag{34}$$

and $\beta \geq 0$ is the multiplexing gain of the cooperative link, *i.e.*,

$$C_{12} \triangleq \beta C(P). \tag{35}$$

Here, the GDOF of the proposed scheme with rate constraints (13-15) is derived.

Using the power allocation (4) and Eq. (34), a can be written as $a = P^{\alpha-1}$. As mentioned earlier, optimizing the achievable sum-rate of Eq. (17) requires finding optimum power allocation parameter γ, which is mathematically involved. To characterize the GDOF, the sum-rate from Eq. (17) for each region $\mathcal{B}_i$ is calculated assuming a suboptimal fixed γ, for $P \to \infty$. Then the minimum of the sum-rates is considered for GDOF analysis. It is clear that this assumption on γ provides an achievable GDOF that might be smaller than the GDOF obtained using the optimal γ. However, by deriving the GDOF associated with the considered upper bounds, it is easy to see that the assumption does not entail any GDOF loss. In fact the GDOF associated with the proposed scheme is the optimal GDOF for the SICUC. Theorem 3 provides the GDOF of the SICUC:

Theorem 3 The optimal GDOF for the SICUC setup is:

$$d(\alpha,\beta) = \begin{cases} 2-2\alpha+\min\big\{\alpha,\beta\big\}, & \alpha < \frac{1}{2}\\ \min\big\{2-\alpha, 2\alpha+\min\{\alpha,\beta\}\big\}, & \frac{1}{2} \leq \alpha < \frac{2}{3}\\ 2-\alpha, & \frac{2}{3} \leq \alpha < 1\\ \alpha, & 1 \leq \alpha < 2\\ \min\big\{2+\beta,\alpha\big\}. & 2 \leq \alpha \end{cases} \tag{36}$$

Proof: The derivation involves straightforward computation, and hence is omitted. ■

It is remarked that if $\beta = 0$, the same GDOF for the symmetric IC derived in [5] will be obtained.

Figures 2 and 3 show the GDOF as a function of α and β. Fig. 2 gives the GDOF for $\beta \leq \frac{1}{2}$. Compared to the GDOF of the IC, one can realize that increasing the amount of cooperation (corresponding to the value of β) is beneficial in terms of the GDOF for the ranges $\alpha < \frac{2}{3}$ and $\alpha > 2$. For instance, it increases the GDOF from $2-2\alpha$ to $2-\alpha$ for $\alpha \leq \frac{1}{2}$ and $\alpha \leq \beta$ compared to the IC case. Fig. 3 demonstrates the GDOF for $\frac{1}{2} \leq \beta$. Here, increasing the amount of cooperation is not useful for $\alpha \leq 2$. For the purpose of comparison, the GDOF of the two known extreme cases of $\beta = 0$ *i.e.*, the IC [5] and $\beta = \infty$, *i.e.*, the CgRC [9] are plotted as well.

VI. Signalling for $aP \leq 1$

The interference-limited regime, *i.e.*, $aP \geq 1$ has so far been considered in this paper. In this section, the sum-rate analysis is done for the case that noise is the major performance-limiting factor in the communication system shown in Fig. 1.

Theorem 4 If $aP \leq 1$, setting $P_u = P$, *i.e.*, treating the interference as noise, achieves within 1 bit of the sum-capacity of the channel.

Proof: For the IC, *i.e.*, $C_{12} = 0$, [5] proves the theorem. The achievable sum-rate becomes:

$$R^{\text{nl}}_{\text{sum}} = \log(1+\frac{P}{1+aP}). \tag{37}$$

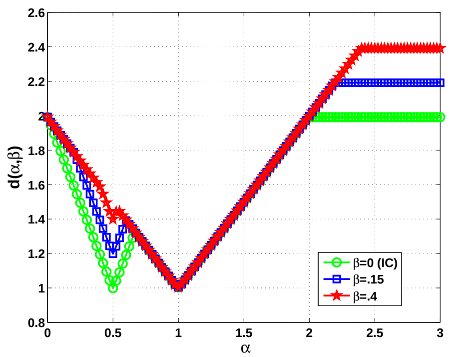

Fig. 2. The effect of partial cooperation on the GDOF of the SICUC for $\beta < \frac{1}{2}$.

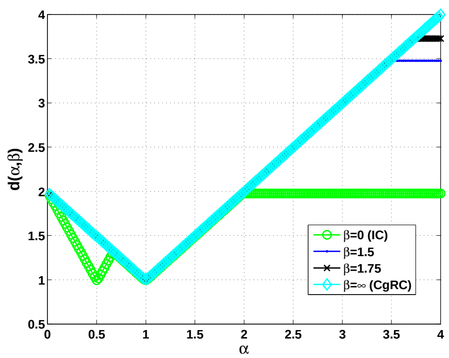

Fig. 3. The effect of partial cooperation on the GDOF of the SICUC for $\frac{1}{2} \leq \beta$. $\beta = 0$ is also considered for the purpose of comparison.

To extend the result of [5], to $C_{12} > 0$ case, this rate is compared to the upper bounds obtained from CgRC.

For $a \leq 1$, the difference $\Delta^{\text{nl}}_{a\leq 1}$ between the achievable sum-rate of Eq. (37) and the upper bound (22) becomes:

$$\begin{aligned}
\Delta^{\text{nl}}_{a\leq 1} &= \frac{1}{2}\Big[\log\big(1+(1+\sqrt{a})^2P\big)+\log\big(\frac{1+P}{1+aP}\big)- \\
&\quad -2\log\big(\frac{1+aP+P}{1+aP}\big)\Big] \\
&= \frac{1}{2}\Big[\log\big(\frac{1+(1+\sqrt{a})^2P}{1+aP+P}\big)+\log\big(\frac{(1+P)(1+aP)}{1+aP+P}\big)\Big] \\
&= \frac{1}{2}\Big[\log\big(1+\frac{2\sqrt{a}P}{1+aP+P}\big)+\log\big(1+\frac{aP^2}{1+aP+P}\big)\Big] \\
&\overset{(a)}{\leq} \frac{1}{2}\Big[\log\big(1+\frac{aP+P}{1+aP+P}\big)+\log\big(1+\frac{P}{1+aP+P}\big)\Big] \\
&\leq 1,
\end{aligned}$$

where (a) follows from $2\sqrt{a} \leq (1+a)$ and $aP \leq 1$.

Similarly, by comparing the achievable sum-rate of Eq. (37) and the second term in the upper bound of Eq. (31), it can be shown that the gap is not greater than 1 bit for $a > 1$ as follows:

$$\begin{aligned}
\Delta^{\text{nl}}_{a>1} &= \frac{1}{2}\log\big(1+(1+\sqrt{a})^2P\big)-\log\big(1+\frac{P}{1+aP}\big) \\
&= \frac{1}{2}\Big[\log\big(\frac{1+(1+\sqrt{a})^2P}{1+aP+P}\big)+\log\big(\frac{(1+aP)^2}{1+aP+P}\big)\Big] \\
&= \frac{1}{2}\Big[\log\big(1+\frac{2\sqrt{a}P}{1+aP+P}\big)+\log\big(1+\frac{(aP)^2+aP-P}{1+aP+P}\big)\Big] \\
&\overset{(a)}{\leq} \frac{1}{2}\Big[\log\big(1+\frac{aP+P}{1+aP+P}\big)+\log\big(1+\frac{1+aP-P}{1+aP+P}\big)\Big] \\
&\leq \frac{1}{2}[1+1], \\
&\leq 1,
\end{aligned}$$

where in (a), $2\sqrt{a}$ and $(aP)^2$ are replaced by the larger quantities $(1+a)$ and 1, respectively.

■

VII. Conclusion

We achieved within two bits of the sum-capacity of the symmetric interference channel with unidirectional cooperation by applying simple HK scheme in conjunction with cooperative communication. In particular, we proposed a power allocation strategy to divide power between private, common and cooperative codewords for each transmitter in the weak interference regime. For the strong interference regime we simplified the achievablity scheme to use one or both of common and cooperative codebooks for different scenarios. The simplified schemes ensure the maximum gap of 1 bit from the sum-capacity of the channel.

Appendix A
Proof of Theorem 1

The proof consists of the following steps:

1) $R_1 - R_i \leq \frac{1}{2}$ for $i \in \{3,4,5\}$,

2) $\Delta_1 \triangleq R^{(1)}_{\text{ub}} - R'_1 \leq 1.2$,

3) $\Delta_2 \triangleq R^{(2)}_{\text{ub}} - R'_2 \leq 2$,

where $R_1, \cdots, R_5$ are defined in Eq. (20), and

$$R'_i \triangleq R_i + 2C(\frac{P_u}{2}) \quad \text{for} \quad i \in \{1,2\}.$$

To prove the first step, it is sufficient to show

$$\Delta' \triangleq C(\frac{(2+a)P_w}{P_u+2}) - C(\frac{(1+a)P_w}{P_u+2}) \leq \frac{1}{2},$$

which is true since:

$$\Delta' = \frac{1}{2}\log(1+\frac{P_w}{P_w(1+a)+P_u+2}) \leq \frac{1}{2},$$

where b is defined in Eq. (26). R'_1 and R'_2 can be written as:

$$\begin{aligned}
R'_1 &= \frac{1}{2}\log\Big(\frac{(2a+1)\big[ab(1+(2+a)P)+a-2\sqrt{a}\big]}{4a^2(1+b)}\Big), \\
R'_2 &= C_{12}+\log\Big(\frac{(2a+1)+b(1+a+a^2P)}{a(1+b)}\Big).
\end{aligned}$$

We define:

$$\begin{aligned} X &\triangleq abP \\ A &\triangleq 1+b \\ B &\triangleq aA \\ C &\triangleq (2+a)(2a+1) \\ D &\triangleq (2a+1)(ab+a-2\sqrt{a}). \end{aligned}$$

Hence:

$$\begin{aligned} \Delta_1 &= 1+\frac{1}{2}\big[\log(\frac{AX+B}{CX+D})+\log(\frac{aP+a}{aP+1})\big] \\ &\overset{\text{(I)}}{\leq} 1+\frac{1}{2}\log(\frac{AX+B}{CX+D}) \\ &\overset{\text{(II)}}{\leq} 1+\frac{1}{2}\log(\frac{Ab+B}{Cb+D}) \\ &\overset{\text{(III)}}{\leq} 1.2 \end{aligned}$$

Since $a \leq 1$ (I) is true. Because $AD-BC \leq 0$, $\frac{AX+B}{CX+D}$ is a monotonically decreasing function of X for $\frac{-D}{C} \leq X$. Noting that $aP \geq 1$, it is straightforward to show $\frac{-D}{C} \leq b \leq X$, therefore (II) is also true. As $\frac{Ab+B}{Cb+D}$ is only a function of a, (III) can be simply verified. In addition:

$$\Delta_2 = 1+\log\Big(\frac{a(b+1)\big[P+(aP+1)^2\big]}{(aP+1)\big[b(a+1+a^2P)+2a+1\big]}\Big),$$

and it is easy to show that $\Delta_2 \leq 2$. Therefore, the gap is at most 2 bits which occurs when $R=R_2$, where R is defined in Eq. (19).

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